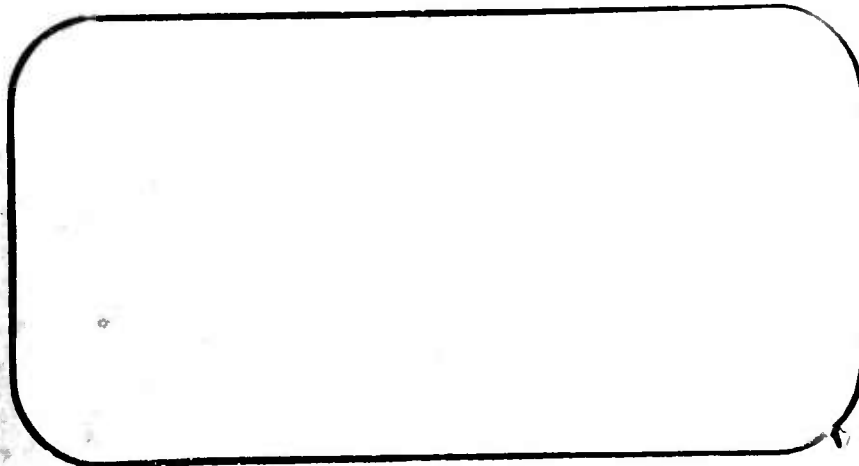


AD 668920

DEPARTMENT  
OF  
OPERATIONS RESEARCH



DDC  
RECEIVED  
MAY 16 1968  
RECEIVED



COLLEGE OF ENGINEERING  
CORNELL UNIVERSITY  
ITHACA, NEW YORK 14850

This document has been approved  
for public release and sale; its  
distribution is unlimited

Reproduced by the  
CLEARINGHOUSE  
for Federal Scientific & Technical  
Information Springfield Va 22151

DEPARTMENT OF OPERATIONS RESEARCH  
COLLEGE OF ENGINEERING  
CORNELL UNIVERSITY  
ITHACA, NEW YORK

TECHNICAL REPORT NO. 42

February 1968

MULTIPLE COMPARISONS WITH A CONTROL  
FOR MULTIPLY-CLASSIFIED VARIANCES  
OF NORMAL POPULATIONS

by

Robert E. Bechhofer

Prepared under Contracts

DA-31-124-ARO-D-474, U.S. Army Research Office-Durham

and

Nonr-401(53), Office of Naval Research

Reproduction in Whole or in Part is Permitted for  
any Purpose of the United States Government

MULTIPLE COMPARISONS WITH A CONTROL  
FOR MULTIPLY-CLASSIFIED VARIANCES OF NORMAL POPULATIONS\*

Robert E. Bechhofer

Department of Operations Research, Cornell University

and

Statistical Laboratory, University of Cambridge

Abstract

Dunnett has prepared tables for making multiple comparisons with a control for single-factor (or multi-factor) experiments involving means of normal populations. In the present paper the author uses the multiplicative model for variances which he introduced in an earlier paper, as the basis for making multiple comparisons with a control for multi-factor experiments involving variances of normal populations. A limited set of tables for use with this procedure is included.

---

\*This research was supported by the U.S. Army Research Office-Durham under Contract DA-31-124-ARO-D-474 and the Office of Naval Research under Contract N-onr-401(53). Reproduction in whole or in part is permitted for any purpose of the United States Government.

## 1. Introduction

In [1] it was shown how to design experiments for ranking multiply-classified variances of normal populations. For such experiments there will, of course, be situations in which ranking of the "effects" is not the objective of the experimenter, but rather he would like to estimate these effects. The purpose of the present note is to show how the theory developed in [1] provides a direct method of making certain joint confidence interval estimates of the effects. We shall use the notation of [1], and in particular that of Section 2.

## 2. Percentage points for joint interval estimation

For fixed  $a \geq 2$ ,  $b \geq 1$ ,  $n \geq 1$ , and for  $i = 2, 3, \dots, a$  we define the  $a-1$  chance variables

$$(1) \quad G_n^{(i),b} = \prod_{j=1}^b \chi_n^2(i,j) / \chi_n^2(1,j) ,$$

where the  $\chi_n^2(i,j)$  ( $1 \leq i \leq a$ ;  $1 \leq j \leq b$ ) are  $a \cdot b$  independent chi-square variates, each based on  $n$  d.f. (Thus for fixed  $i$ ,  $G_n^{(i),b}$  is the product of  $b$  independent central F-statistics, each based on  $(n,n)$  d.f.; for  $i_1 \neq i_2$ ,  $G_n^{(i_1),b}$  and  $G_n^{(i_2),b}$  are correlated.) For  $0 < \lambda < 1$  let  $G_n^{a,b}(1-\lambda)$  be defined by the equation

$$(2) \quad P(G_n^{a,b}(1-\lambda) < G_n^{(i),b} \quad (i = 2, 3, \dots, a)) = 1-\lambda .$$

We now show how the  $G_n^{a,b}(1-\lambda)$  can be used to make certain types of inferences concerning the effects  $\alpha_i$  defined by (2.1) of [1] when (2.2) holds. As in Section 3.3 of [1], we let  $\hat{s}_{ij}^2$  denote the sample variance based on  $n$  d.f. associated with the  $i^{\text{th}}$  "level" of Factor A and the  $j^{\text{th}}$  "level" of Factor B.

### 3. Confidence statements

#### 3.1 Multiple comparisons with a control

Suppose that the first level of Factor A is a "control" level, and that the remaining are "test" levels, i.e., that  $\alpha_1$  is the effect associated with the control level, and that  $\alpha_2, \alpha_3, \dots, \alpha_a$  are the effects associated with the test levels. Then it may be desired to make a joint confidence statement concerning the effect-ratios  $\alpha_i/\alpha_1$  ( $i=2,3,\dots,a$ ). Clearly the following statement holds with confidence coefficient  $1-\lambda$ :

$$(3) \quad \left\{ \frac{\alpha_i}{\alpha_1} < \left[ \frac{1}{G_n^{a,b}(1-\lambda)} \prod_{j=1}^b \frac{\hat{s}_{ij}^2}{\hat{s}_{1j}^2} \right]^{1/b} \quad (i=2,3,\dots,a) \right\}.$$

This is sometimes referred to as a (one-sided) multiple comparison with a control, and is the analogue for multiply-classified variances of [2] (wherein  $b=1$ ) for means. (See also [3] (wherein  $b=1$ ) for the corresponding two-sided comparisons for means.)

#### 3.2 Two-sided estimates when $a=2$ .

When  $a=2$  and  $(0 \leq \lambda_1, \lambda_2; \lambda_1 + \lambda_2 < 1)$ , the following statement holds with confidence coefficient  $1 - \lambda_1 - \lambda_2$ :

$$(4) \quad \left\{ \left[ \frac{1}{G_n^{2,b}(\lambda_2)} \prod_{j=1}^b \frac{\hat{s}_{2j}^2}{\hat{s}_{1j}^2} \right]^{1/b} < \frac{\alpha_2}{\alpha_1} < \left[ \frac{1}{G_n^{2,b}(1-\lambda_1)} \prod_{j=1}^b \frac{\hat{s}_{2j}^2}{\hat{s}_{1j}^2} \right]^{1/b} \right\}.$$

This is a two-sided interval estimate of  $\alpha_2/\alpha_1$ .

#### 4. Evaluation of the constants $G_n^{a,b}(\lambda)$

Comparison of (2) with (4.1) and (6.2) of [1] shows that

$$(5a) \quad P_A(n|a,b; \theta_A^*) = 1-\lambda$$

$$(5b) \quad (1/\theta_A^*)^b = G_n^{a,b}(1-\lambda).$$

Thus, for fixed  $a \geq 2$ ,  $b \geq 1$ ,  $n \geq 1$  and specified  $\lambda$ , one can determine  $\theta_A^*$  uniquely from (5a); this value of  $\theta_A^*$  substituted in (5b)

then yields  $G_n^{a,b}(1-\lambda)$ . Clearly, for  $a=2$  we have

$$1/G_n^{2,b}(1-\lambda) = G_n^{2,b}(\lambda) = (\theta_A^*)^b.$$

We have used these relationships and the formulae of Section 4.1 of [1] to determine exact values of  $G_n^{2,b}(\lambda)$  for  $\lambda = 0.25, 0.15, 0.10, 0.05, 0.025, 0.01, 0.005, 0.001, 0.0005$ ; Table I gives the G-values for  $b=2$  and  $n=1(1)6,8$  while Table II gives the G-values for  $b=3$  and  $n=2,4$ . These percentage points are correct to the number of significant figures which are tabulated. The tables can be regarded as extensions of the tables [4]

Table I

Table of exact upper percentage points,  $G_n^{2,2}(\lambda)$ , of the  
product of two independent central F-statistics,  
each based on (n,n) degrees of freedom

$$P\{G_n^{2,2}(\lambda) < G_n^{(2),2}\} = \lambda$$

$\lambda \backslash n$	0.25	0.15	0.10	0.05	0.025	0.01	0.005	0.001	0.0005
1	15.081	72.626	226.91	1405.4	7917.0	71,272	35,941x10	13,983x10 <sup>3</sup>	65,707x10 <sup>3</sup>
2	5.1157	12.746	24.259	66.115	166.94	529.35	1224.2	7987.8	17,547
3	3.5038	6.9963	11.321	23.697	46.344	105.42	189.98	696.74	1195.2
4	2.8633	5.0940	7.5856	13.905	23.969	46.361	74.034	205.43	312.57
5	2.5166	4.1627	5.8860	9.9441	15.878	27.873	41.433	97.585	138.41
6	2.2970	3.6105	4.9246	7.8653	11.921	19.605	27.781	58.709	79.514
8	2.0310	2.9817	3.8773	5.7520	8.1480	12.322	16.433	30.312	38.767

Table II

Table of exact upper percentage points,  $G_n^{2,3}(\lambda)$ , of the  
product of three independent central F-statistics,  
each based on (n,n) degrees of freedom

$$P\{G_n^{2,3}(\lambda) < G_n^{(2),2}\} = \lambda$$

$\lambda \backslash n$	0.25	0.15	0.10	0.05	0.025	0.01	0.005	0.001	0.0005
2	7.6601	23.570	51.557	171.46	510.39	1939.4	5039.3	41,118	97,971
4	3.6712	7.4496	12.111	25.224	48.422	105.71	182.76	592.34	954.27

when the d.f.  $(v_1, v_2)$  of those tables are such that  $v_1 = v_2 = n$ .  
 (Thus the diagonal entries in the tables [4] are the percentage points  $G_n^{2,1}(\lambda)$ .)

Additional exact values of  $G_n^{a,b}(\lambda)$  for  $a > 1, b, n$  small, and  $\lambda$  close to zero must await the evaluation of the integral (4.4) of [1] to obtain formulae like those of Section 4.1 of [1].

For  $a=2$ , Tables I and II can also be used for size and power calculations associated with the test  $H_0: \alpha_1 = \alpha_2$  vs. a one-sided alternative  $H_1: \alpha_1 > \alpha_2$  (say).



- [1] Bechhofer, R.E.: "Single-stage procedures for ranking multiply-classified variances of normal populations," Technometrics, Vol. 10 (1968) pp.
- [2] Dunnett, C.W.: "A multiple comparison procedure for comparing several treatments with a control," Journal of the American Statistical Association, Vol. 50 (1955), pp. 1096-1121.
- [3] Dunnett, C.W.: "New tables for multiple comparisons with a control," Biometrics, Vol. 20 (1964), pp. 482-491.
- [4] Merrington, M. and Thompson, C.M.: "Tables of percentage points of the inverted beta (F) distribution," Biometrika, Vol. 33 (1943), pp. 73-88.

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D

*(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)*

1. ORIGINATING ACTIVITY (Corporate author) Department of Operations Research College of Engineering, Cornell University Ithaca, New York 14850		2a. REPORT SECURITY CLASSIFICATION <b>Unclassified</b>	
		2b. GROUP	
3. REPORT TITLE  MULTIPLE COMPARISONS WITH A CONTROL FOR MULTIPLY-CLASSIFIED VARIANCES OF NORMAL POPULATIONS			
4. DESCRIPTIVE NOTES (Type of report and inclusive dates) <b>Technical Report, February 1968</b>			
5. AUTHOR(S) (First name, middle initial, last name)  Robert E. Bechhofer			
6. REPORT DATE <b>February 1968</b>		7a. TOTAL NO. OF PAGES <b>6 + 1</b>	7b. NO. OF REFS <b>4</b>
8a. CONTRACT OR GRANT NOS. <b>DA-31-124-ARO-D-474</b> b. <del>XXXXXXXXXX</del> <b>Nonr-401(53)</b> c. d.		9a. ORIGINATOR'S REPORT NUMBER(S) <b>Technical Report No. 42</b>  9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	
10. DISTRIBUTION STATEMENT  Distribution of this document is unlimited			
11. <del>XXXXXXXXXXXXXXXXXXXX</del> Sponsoring military activity U.S. Army Research Office- Durham, N.C. 27706		12. SPONSORING MILITARY ACTIVITY <b>Logistics and Mathematical Statistics Branch, Office of Naval Research Washington, D.C. 20360</b>	
13. ABSTRACT  Dunnett has prepared tables for making multiple comparisons with a control for single-factor (or multi-factor) experiments involving means of normal populations. In the present paper the author uses the multiplicative model for variances which he introduced in an earlier paper, as the basis for making multiple comparisons with a control for multi-factor experiments involving variances of normal populations. A limited set of tables for use with this procedure is included.			

DD FORM 1473  
1 NOV 65

Unclassified

Security Classification

Unclassified

Security Classification

14	KEY WORDS		LINK A		LINK B		LINK C	
			ROLE	WT	ROLE	WT	ROLE	WT
Mathematical statistics Multiple comparisons with a control Variances Factorial experiments F-test of homogeneity of variances Confidence intervals								

Unclassified

Security Classification